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Stress analysis of an unbounded elastic solid with orthotropic inclusions and voids using a new integral equation technique

Jungki Lee [a], Sungjoon Choi [a], Ajit Mal [b,*]

[a] *Department of Mechano-Informatics and Design Engineering, Hongik University, Jochiwon-Eup, Yeonki-Gun, Chungnam, 339-701 South Korea*

[b] *Department of Mechanical and Aerospace Engineering, Eng IV, Room 48-121, School of Engineering and Applied Science, University of California, Los Angeles, CA 90095-1597, USA*

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Abstract

A recently developed numerical method based on a volume integral formulation is applied to calculate the elastostatic field in an unbounded isotropic elastic medium containing orthotropic inclusions subject to remote loading. A modified form of the method in which the integral equations involve volumes of the inclusions and boundaries of voids or cracks is used to deal with the presence of both types of inhomogeneity. A detailed analysis of displacement and stress fields is carried out for orthotropic cylindrical and elliptic cylindrical inclusions as well as voids. The accuracy and effectiveness of the new methods are examined through comparison with results obtained from analytical and boundary integral equation methods. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The calculation of the stress and strain fields in solids containing multiple inclusions and voids or cracks and subjected to external loads is of considerable interest in a variety of engineering applications. A notable example is the stress analysis of damaged fiber reinforced composites that consist of a large number of densely packed fibers with voids or cracks in the matrix. The matrix and the fibers in composites are usually made of isotropic material. However, some of the constituents can be anisotropic. As an example, in SiC/Ti metal matrix composites, the matrix is nearly isotropic, but the SiC fibers have strong anisotropy. Structural composites are often subject to manufacturing and/or service induced defects that strongly affect the remaining life of the structure. A precise knowledge of the deformation and stress fields near interacting isotropic or anisotropic fibers and voids/microcracks under remote loading can be extremely helpful in predicting the failure and damage mechanisms in the composites. Several techniques have been proposed for analyzing multiple-inclusion interactions in an infinite medium (Mal and Yang, 1994; Chen, 1993;

* Corresponding author. Tel.: +1-310-825-5481; fax: +1-310-206-4830.
E-mail address: ajit@ucla.edu (A. Mal).

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Duan et al., 1986; Johnson et al., 1980; Moschovidis and Mura, 1975; McPedran and Movchan, 1994; Kushch, 1998). However, none of these methods can be applied to the general problem in which the inclusions or voids are of arbitrary shape and their concentration is high. To our knowledge, the only available methods to solve problems of this type are the finite element (FEM) or the boundary integral equation (BIEM) method. However, the FEM is most effective when the domain of the problem is finite and it is often not possible to separate the influence of the boundary from that of the "microscopic" features of the material on the elastic field. Conventional FEMs cannot be directly applied to infinite domains. The BIEM is, in principle, applicable to this class of problems since it can be applied to infinite domains. However, since Green's function for anisotropic inclusions is involved in the BIEM and Green's function for an anisotropic material is much more complex than that for isotropic materials, their numerical treatment of the BIEs becomes extremely cumbersome (Lee and Mal, 1990).

In this paper, a volume integral equation method (VIEM) and a mixed volume and boundary integral equation method are developed for the effective and accurate calculation of the stresses and displacements in unbounded isotropic solids in the presence of multiple anisotropic inclusions and voids or cracks. It should be noted that these newly developed numerical methods do not require Green's function for anisotropic inclusions, and they can also be applied to general two-dimensional elastostatic as well as elastodynamic problems for arbitrary geometry and a number of inhomogeneities. In the formulation of the methods, the continuity condition at each interface is automatically satisfied, and in contrast to FEM, where the full domain needs to be discretized, these methods require discretization of the inclusions only. Finally, the methods take full advantage of the pre- and post-processing capabilities developed in FEM and BIEM.

In this paper, a detailed analysis of the displacement and stress fields are carried out for an unbounded isotropic matrix containing orthotropic cylindrical and elliptic cylindrical inclusions and voids. The accuracy and effectiveness of the new methods are examined through comparison with results obtained from analytical and boundary integral equation methods. It is demonstrated that these new methods are very accurate and effective for solving plane elastostatic problems in unbounded solids containing anisotropic inclusions and voids.

2. The integral equations

2.1. The volume integral equation method

The geometry of the general elastostatic problem is shown in Fig. 1. Let $c_{ijkl}^{(1)}$ denote the elastic stiffness tensor of the inclusion and $c_{ijkl}^{(2)}$ those of the unbounded matrix material. The matrix is assumed to be homogeneous and isotropic so that $c_{ijkl}^{(2)}$ is a constant isotropic tensor, while $c_{ijkl}^{(1)}$ is arbitrary, i.e., the inclusions may, in general, be inhomogeneous and anisotropic. The interfaces between the inclusions and the matrix are assumed to be perfectly bonded insuring the continuity of the displacement and stress vectors. The elastostatic VIE is given by (Mal and Knopoff, 1967; Lee and Mal, 1997)

$$u_m(\mathbf{x}) = u_m^{\mathrm{o}}(\mathbf{x}) - \int_R \delta c_{ijkl} g_{i,j}^m(\boldsymbol{\xi}, \mathbf{x}) u_{k,l}(\boldsymbol{\xi}) \mathrm{d}\xi, \tag{1}$$

where the integral is over whole space, $\delta c_{ijkl} = c_{ijkl}^{(1)} - c_{ijkl}^{(2)}$, and $g_i^m(\boldsymbol{\xi}, \mathbf{x})$ is the static Green's function (or Kelvin's solution) for the unbounded matrix material, i.e., $g_i^m(\boldsymbol{\xi}, \mathbf{x})$ represents the ith component of the displacement at $\boldsymbol{\xi}$ due to unit concentrated force at $\mathbf{x}$ in the mth direction. In Eq. (1), the summation convention and comma notation have been used and the differentiations are with respect to ξ_i. It should be noted that the integrand is nonzero within the inclusions only, since $\delta c_{ijkl} = 0$, outside the inclusions.

If $\mathbf{x} \in \mathrm{R}$, then Eq. (1) is an integrodifferential equation for the unknown displacement vector $\mathbf{u}(\mathbf{x})$, it can, in principle, be determined through the solution of Eq. (1). An algorithm based on the discretization of

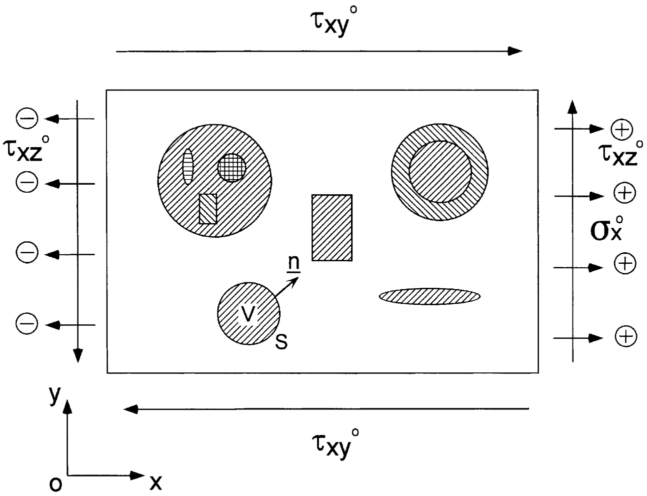


Fig. 1. Geometry of the general elastostatic problem.

Eq. (1) was developed by Lee and Mal (1995, 1997) to determine the unknown displacement vector $\mathbf{u}(\mathbf{x})$ by discretizing the inclusions using standard finite elements. Once $\mathbf{u}(\mathbf{x})$ within the inclusions is determined, the displacement field outside the inclusions can be obtained from Eq. (1) by evaluating the integral, and the stress field within and outside the inclusions can also be determined without any difficulty. The details of the numerical treatment of Eq. (1) can be found in Lee and Mal (1995, 1997) and will be omitted. In what follows the formulation for plane strain, elastostatic displacement and stress fields is developed for an unbounded isotropic matrix containing orthotropic inclusions as well as voids.

Let the coordinate axes x_1, x_2, x_3 be taken parallel to the symmetry axes of the orthotropic material. Under plane strain assumption in the 1–2 plane, the constitutive relation for the orthotropic inclusions can be expressed in the form,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix}. \tag{2}$$

The stiffness constants c_{ij} are related to the engineering moduli through

$$\begin{aligned} c_{11} &= \frac{E_1(1-v_{23}v_{32})}{\Delta}, \qquad c_{12} = \frac{E_2(v_{12}+v_{32}v_{13})}{\Delta} = \frac{E_1(v_{21}+v_{23}v_{31})}{\Delta}, \\ c_{22} &= \frac{E_2(1-v_{31}v_{13})}{\Delta}, \qquad c_{66} = \mu_{66}, \end{aligned} \tag{3}$$

where

$$\Delta = 1 - v_{23}v_{32} - v_{13}v_{31} - v_{12}v_{21} - 2v_{21}v_{32}v_{13}. \tag{4}$$

In the above equation, E_1, E_2, E_2 are Young's moduli in 1, 2 and 3 directions, respectively, μ_{66} is the shear modulus in the 1–2 plane and v_{ij} is Poisson's ratio for transverse strain in the *j*-direction when stressed in the *i*-direction. Thus, for uniaxial stress along x_i,

$$v_{ij} = -\frac{\epsilon_j}{\epsilon_i}. \tag{5}$$

It should be noted that the elastic moduli satisfy the reciprocal relationships $v_{ij}/E_i = v_{ji}/E_j$.

For plane strain problems, the volume integral equation (1) becomes

$$u_1(\mathbf{x}) = u_1^{\mathrm{o}}(\mathbf{x}) - \int_R \{[\delta c_{11} g^1_{1,1} u_{1,1} + \delta c_{12} g^1_{1,1} u_{2,2} + \delta c_{66} g^1_{1,2}(u_{1,2} + u_{2,1})] + [\delta c_{22} g^1_{2,2} u_{2,2} + \delta c_{12} g^1_{2,2} u_{1,1} + \delta c_{66} g^1_{2,1}(u_{1,2} + u_{2,1})]\} \mathrm{d}\xi_1 \mathrm{d}\xi_2, \tag{6}$$

$$u_2(\mathbf{x}) = u_2^{\mathrm{o}}(\mathbf{x}) - \int_R \{[\delta c_{11} g^2_{1,1} u_{1,1} + \delta c_{12} g^2_{1,1} u_{2,2} + \delta c_{66} g^2_{1,2}(u_{1,2} + u_{2,1})] + [\delta c_{22} g^2_{2,2} u_{2,2} + \delta c_{12} g^2_{2,2} u_{1,1} + \delta c_{66} g^2_{2,1}(u_{1,2} + u_{2,1})]\} \mathrm{d}\xi_1 \mathrm{d}\xi_2, \tag{7}$$

where $u_1(\mathbf{x})$, $u_2(\mathbf{x})$ are the in-plane displacement components and $\delta c_{11} = c_{11} - (\lambda + 2\mu)$, $\delta c_{12} = c_{12} - \lambda$, $\delta c_{22} = c_{22} - (\lambda + 2\mu)$, $\delta c_{66} = c_{66} - \mu$, and λ, μ are the Lamé constants for the isotropic matrix material.

In Eqs. (6) and (7), g_i^m is Green's function for the unbounded isotropic matrix material. Thus, the VIEM does not require the use of Green's function for the anisotropic inclusions. This is in contrast to the BIEM, where Green's functions for both the matrix and the inclusions are involved in the formulation of the equations. For uniformly applied remote stress, the displacement vector $\mathbf{u}^{\circ}$ is of the form,

$$u_1^{\mathrm{o}} = C_1 x, \qquad u_2^{\mathrm{o}} = C_2 y, \qquad u_3^{\mathrm{o}} = C_3 x + C_4 y, \tag{8}$$

where the constants C_1–C_4 are related to the tensile and shear components of the applied stress.

2.2. The boundary integral equation method

The integral equation on the outer surface S_+ of the anisotropic inclusion can be expressed as (Banerjee, 1993; Rizzo et al., 1985)

$$u_m(\mathbf{x}) = u_m^{\mathrm{o}}(\mathbf{x}) + c_{ijkl}^{(\mathrm{M})} \int_{S_+} \left[g_{k,l}^{m(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) u_i(\boldsymbol{\xi}) - g_i^{m(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) u_{k,l}(\boldsymbol{\xi}) \right] n_j \,\mathrm{d}S, \tag{9}$$

while for the interior surface S_-,

$$u_m(\mathbf{x}) = -c_{ijkl}^{(\mathrm{I})} \int_{S_-} \left[g_{k,l}^{m(\mathrm{I})}(\boldsymbol{\xi}, \mathbf{x}) u_i(\boldsymbol{\xi}) - g_i^{m(\mathrm{I})}(\boldsymbol{\xi}, \mathbf{x}) u_{k,l}(\boldsymbol{\xi}) \right] n_j \,\mathrm{d}S. \tag{10}$$

In Eqs. (9) and (10), $\mathbf{n}$ is the outward unit normal to S_+, and superscripts (M) and (I) indicate that the quantities involved are for the isotropic matrix and the inclusions, respectively. Eqs. (9) and (10), together with the continuity conditions across S, give rise to the BIE for $\mathbf{u}(\mathbf{x})$. When the inclusion becomes a void, the integral equations reduce to the standard BIE,

$$u_m(\mathbf{x}) = u_m^{\mathrm{o}}(\mathbf{x}) + c_{ijkl}^{(\mathrm{M})} \int_S g_{k,l}^{m(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) u_i(\boldsymbol{\xi}) n_j \,\mathrm{d}S. \tag{11}$$

The integral equations on the outer surface of the orthotropic inclusion can be expressed as

$$\begin{aligned} u_1(\mathbf{x}) = u_1^{\mathrm{o}}(\mathbf{x}) - \int_{S_+} \Big[& g_1^{1(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) t_1(\boldsymbol{\xi}) + g_2^{1(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) t_2(\boldsymbol{\xi}) - T_1^{1(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) u_1(\boldsymbol{\xi}) \\ & - T_2^{1(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) u_2(\boldsymbol{\xi}) \Big] \mathrm{d}S(\boldsymbol{\xi}), \\ u_2(\mathbf{x}) = u_2^{\mathrm{o}}(\mathbf{x}) - \int_{S_+} \Big[& g_1^{2(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) t_1(\boldsymbol{\xi}) + g_2^{2(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) t_2(\boldsymbol{\xi}) - T_1^{2(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) u_1(\boldsymbol{\xi}) \\ & - T_2^{2(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) u_2(\boldsymbol{\xi}) \Big] \mathrm{d}S(\boldsymbol{\xi}), \end{aligned} \tag{12}$$

where $g_\alpha^{\beta(\mathrm{M})}$ is Green's function for the unbounded isotropic matrix material and is given by (Banerjee, 1993; Rizzo et al., 1985)

$$g_\alpha^{\beta(\mathrm{M})} = \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}\left[-\frac{\lambda+3\mu}{\lambda+\mu}\ln r\delta_{\alpha\beta} + r_{,\alpha}r_{,\beta}\right], \tag{13}$$

where $r = |\mathbf{x}-\boldsymbol{\xi}|$ and $\alpha,\beta = 1,2$ and λ,μ are the Lamé constants for the unbounded isotropic matrix material, and the associated tractions, $T_\alpha^{\beta(\mathrm{M})}(\alpha,\beta = 1,2)$ and t_α are given by

$$\begin{aligned}
T_1^{1(\mathrm{M})} &= (\lambda+2\mu)g_{1,1}^{1(\mathrm{M})}n_1 + \lambda g_{2,2}^{1(\mathrm{M})}n_1 + \mu\left(g_{1,2}^{1(\mathrm{M})} + g_{2,1}^{1(\mathrm{M})}\right)n_2,\\
T_1^{2(\mathrm{M})} &= (\lambda+2\mu)g_{1,1}^{2(\mathrm{M})}n_1 + \lambda g_{2,2}^{2(\mathrm{M})}n_1 + \mu\left(g_{1,2}^{2(\mathrm{M})} + g_{2,1}^{2(\mathrm{M})}\right)n_2,\\
T_2^{1(\mathrm{M})} &= (\lambda+2\mu)g_{2,2}^{1(\mathrm{M})}n_2 + \lambda g_{1,1}^{1(\mathrm{M})}n_2 + \mu\left(g_{1,2}^{1(\mathrm{M})} + g_{2,1}^{1(\mathrm{M})}\right)n_1,\\
T_2^{2(\mathrm{M})} &= (\lambda+2\mu)g_{2,2}^{2(\mathrm{M})}n_2 + \lambda g_{1,1}^{2(\mathrm{M})}n_2 + \mu\left(g_{1,2}^{2(\mathrm{M})} + g_{2,1}^{2(\mathrm{M})}\right)n_1,
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
t_1 &= (\lambda+2\mu)u_{1,1}n_1 + \lambda u_{2,2}n_1 + \mu(u_{1,2}+u_{2,1})n_2,\\
t_2 &= (\lambda+2\mu)u_{2,2}n_2 + \lambda u_{1,1}n_2 + \mu(u_{1,2}+u_{2,1})n_1.
\end{aligned} \tag{15}$$

For the interior surface, the equations are

$$\begin{aligned}
u_1(\mathbf{x}) &= \int_{S_-}\left[g_1^{1(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x})t_1(\boldsymbol{\xi}) + g_2^{1(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x})t_2(\boldsymbol{\xi}) - T_1^{1(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x})u_1(\boldsymbol{\xi}) - T_2^{1(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x})u_2(\boldsymbol{\xi})\right]\mathrm{d}S(\boldsymbol{\xi}),\\
u_2(\mathbf{x}) &= \int_{S_-}\left[g_1^{2(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x})t_1(\boldsymbol{\xi}) + g_2^{2(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x})t_2(\boldsymbol{\xi}) - T_1^{2(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x})u_1(\boldsymbol{\xi}) - T_2^{2(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x})u_2(\boldsymbol{\xi})\right]\mathrm{d}S(\boldsymbol{\xi}).
\end{aligned} \tag{16}$$

It should be noted that $g_\alpha^{\beta(\mathrm{I})}$ and $T_\alpha^{\beta(\mathrm{I})}$ $(\alpha,\beta = 1,2)$ in Eq. (16) are the Green's function and their associated tractions for the orthotropic inclusions.

The plane strain Green's function for the orthotropic material is given by (Davì and Milazzo, 1996; Banerjee, 1993; Doblare et al., 1990; Snyder and Cruse, 1975; Tomlin and Butterfield, 1974)

$$\begin{aligned}
g_1^{1(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x}) &= -2[A_{11}B_{11}\ln r_1 + A_{12}B_{12}\ln r_2],\\
g_1^{2(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x}) &= -2\mathrm{i}\left[A_{11}B_{21}\tan^{-1}\left(\frac{y_1}{x}\right) + A_{12}B_{22}\tan^{-1}\left(\frac{y_2}{x}\right)\right],\\
g_2^{1(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x}) &= -2\mathrm{i}\left[A_{21}B_{11}\tan^{-1}\left(\frac{y_1}{x}\right) + A_{22}B_{12}\tan^{-1}\left(\frac{y_2}{x}\right)\right] = g_1^{2(\mathrm{I})},\\
g_2^{2(\mathrm{I})}(\boldsymbol{\xi},\mathbf{x}) &= -2[A_{21}B_{21}\ln r_1 + A_{22}B_{22}\ln r_2],
\end{aligned} \tag{17}$$

where A_{ij} and B_{ij} $(i,j = 1,2)$ are complex constants:

$$\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix} = \begin{bmatrix} b_{11}\mu_1^2 + b_{12} & b_{11}\mu_2^2 + b_{12}\\ b_{12}\mu_1 + b_{22}/\mu_1 & b_{12}\mu_2 + b_{22}/\mu_2\end{bmatrix}, \tag{18}$$

$$\begin{bmatrix} B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix} = \begin{bmatrix} \frac{-A_{22}}{\mu_1A_{22}-\mu_2A_{21}}\cdot\frac{\mathrm{i}}{4\pi} & \frac{A_{21}}{\mu_1A_{22}-\mu_2A_{21}}\cdot\frac{\mathrm{i}}{4\pi}\\ \frac{-A_{12}}{A_{11}-A_{12}}\cdot\frac{\mathrm{i}}{4\pi} & \frac{A_{11}}{A_{11}-A_{12}}\cdot\frac{\mathrm{i}}{4\pi}\end{bmatrix}. \tag{19}$$

The constants b_{ij} are the "reduced compliances", related to the moduli through

$$
\begin{aligned}
&b_{11} = \frac{1 - v_{31}v_{13}}{E_1}, \qquad b_{12} = -\frac{v_{21} + v_{31}v_{23}}{E_2} = -\frac{v_{12} + v_{32}v_{13}}{E_1}, \\
&b_{22} = \frac{1 - v_{32}v_{23}}{E_2}, \quad b_{66} = \frac{1}{\mu_{66}}, \quad b_{16} = 0, \quad b_{26} = 0.
\end{aligned} \tag{20}
$$

The other symbols in Eqs. (17)–(19) are given by

$$
\begin{aligned}
&x = x_1 - \xi_1, \\
&y_1 = \tilde{\mu}_1(x_2 - \xi_2),\ y_2 = \tilde{\mu}_2(x_2 - \xi_2), \\
&r_1 = \sqrt{x^2 + y_1^2},\ r_2 = \sqrt{x^2 + y_2^2},
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
&\mu_1 = \mathrm{i}(a_0 + b_0) = \tilde{\mu}_1 \mathrm{i}, \\
&\mu_2 = \mathrm{i}(-a_0 + b_0) = \tilde{\mu}_2 \mathrm{i},
\end{aligned} \tag{22}
$$

where

$$
a_0 = \left(\frac{b_{12} + \frac{b_{66}}{2} - \sqrt{b_{11}b_{22}}}{2b_{11}} \right)^{1/2}, \quad b_0 = \left(\frac{b_{12} + \frac{b_{66}}{2} + \sqrt{b_{11}b_{22}}}{2b_{11}} \right)^{1/2}. \tag{23}
$$

Since $b_{12} + (b_{66}/2) - \sqrt{b_{11}b_{22}} > 0$, $\tilde{\mu}_1$, $\tilde{\mu}_2$ are real and μ_1, μ_2 are purely imaginary.
The tractions, $T_\alpha^{\beta(\mathrm{I})}$ $(\alpha, \beta = 1, 2)$ and t_α, are

$$
\begin{aligned}
T_1^{1(\mathrm{I})} &= -2c_{11}n_1\left\{ \frac{A_{11}B_{11}x}{r_1^2} + \frac{A_{12}B_{12}x}{r_2^2} \right\} - 2\mathrm{i}c_{12}n_1\left\{ \frac{A_{21}B_{11}\tilde{\mu}_1 x}{r_1^2} + \frac{A_{22}B_{12}\tilde{\mu}_2 x}{r_2^2} \right\} \\
&\quad - 2c_{66}n_2\left\{ \left[\frac{A_{11}B_{11}\tilde{\mu}_1 y_1}{r_1^2} + \frac{A_{12}B_{12}\tilde{\mu}_2 y_2}{r_2^2} \right] - \mathrm{i}\left[\frac{A_{21}B_{11}y_1}{r_1^2} + \frac{A_{22}B_{12}y_2}{r_2^2} \right] \right\}, \\
T_1^{2(\mathrm{I})} &= -2\mathrm{i}c_{11}n_1\left\{ \frac{A_{11}B_{21}y_1}{r_1^2} + \frac{A_{12}B_{22}y_2}{r_2^2} \right\} - 2c_{12}n_1\left\{ \frac{A_{21}B_{21}\tilde{\mu}_1 y_1}{r_1^2} + \frac{A_{22}B_{22}\tilde{\mu}_2 y_2}{r_2^2} \right\} \\
&\quad - 2c_{66}n_2\left\{ \left[\frac{A_{21}B_{21}x}{r_1^2} + \frac{A_{22}B_{22}x}{r_2^2} \right] + \mathrm{i}\left[\frac{A_{11}B_{21}\tilde{\mu}_1 x}{r_1^2} + \frac{A_{12}B_{22}\tilde{\mu}_2 x}{r_2^2} \right] \right\}, \\
T_2^{1(\mathrm{I})} &= -2\mathrm{i}c_{22}n_2\left\{ \frac{A_{21}B_{11}\tilde{\mu}_1 x}{r_1^2} + \frac{A_{22}B_{12}\tilde{\mu}_2 x}{r_2^2} \right\} - 2c_{21}n_2\left\{ \frac{A_{11}B_{11}x}{r_1^2} + \frac{A_{12}B_{12}x}{r_2^2} \right\} \\
&\quad - 2c_{66}n_1\left\{ \left[\frac{A_{11}B_{11}\tilde{\mu}_1 y_1}{r_1^2} + \frac{A_{12}B_{12}\tilde{\mu}_2 y_2}{r_2^2} \right] - \mathrm{i}\left[\frac{A_{21}B_{11}y_1}{r_1^2} + \frac{A_{22}B_{12}y_2}{r_2^2} \right] \right\}, \\
T_2^{2(\mathrm{I})} &= -2c_{22}n_2\left\{ \frac{A_{21}B_{21}\tilde{\mu}_1 y_1}{r_1^2} + \frac{A_{22}B_{22}\tilde{\mu}_2 y_2}{r_2^2} \right\} + 2\mathrm{i}c_{21}n_2\left\{ \frac{A_{11}B_{21}y_1}{r_1^2} + \frac{A_{12}B_{22}y_2}{r_2^2} \right\} \\
&\quad - 2c_{66}n_1\left\{ \left[\frac{A_{21}B_{21}x}{r_1^2} + \frac{A_{22}B_{22}x}{r_2^2} \right] + \mathrm{i}\left[\frac{A_{11}B_{21}\tilde{\mu}_1 x}{r_1^2} + \frac{A_{12}B_{22}\tilde{\mu}_2 x}{r_2^2} \right] \right\},
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
&t_1 = c_{11}u_{1,1}n_1 + c_{12}u_{2,2}n_1 + c_{66}(u_{1,2} + u_{2,1})n_2, \\
&t_2 = c_{22}u_{2,2}n_2 + c_{12}u_{1,1}n_2 + c_{66}(u_{1,2} + u_{2,1})n_1.
\end{aligned} \tag{25}
$$

As the observation point approaches the interface, $u_i(\mathbf{x})$ in Eqs. (12) and (16) are replaced by $(1/2)u_i(\mathbf{x})$ for a smooth interface, and the derivatives of Green's function must be interpreted in the Cauchy principal value sense. Applying the continuity conditions at the interface, followed by discretization, a system of four coupled algebraic equations for the interface displacements and tractions are obtained, which can be solved to give the displacements and stress components at the nodal points of the interface. Once the displacements and tractions at the interface are known, the displacements and stresses everywhere can be calculated by evaluating the integral expressions in Eqs. (12) and (16).

2.3. Numerical formulation

The integral equations (6), (7), (12) and (16) must be solved numerically through discretization of their respective domains and evaluation of the integrals. Clearly, the integrands contain singularities of different orders due to the singular nature of the Green's function at $\mathbf{x} = \xi$ (i.e., $r = 0$), and the evaluation of the integrals requires special attention. The order of the singularity for the isotropic matrix is $\ln r$ in the Green's function and $1/r$ in its derivatives while that for the orthotropic material is $\ln r_\alpha$ in the Green's function and $1/r_\alpha$ in its derivatives where r_α is given in Eq. (21). It should be noted that the VIE involves only $g_\alpha^{\beta(\mathrm{M})}$ and $T_\alpha^{\beta(\mathrm{M})}$ for the isotropic matrix while the BIE involves $g_\alpha^{\beta(\mathrm{I})}$ and $T_\alpha^{\beta(\mathrm{I})}$ for the anisotropic inclusions in addition to these. Furthermore, the singularities in VIEM are weaker (integrable) than those in BIEM, where they are of the Cauchy type. We have used the direct integration scheme as introduced by Cerrolaza and Alarcon (1989), Li and Han (1985) and Lu and Ye (1991), after suitable modifications, to handle the singularities; a description of the modified method used in the discretization of the VIE is given by Lee and Mal (1997).

3. Inclusion problems

In order to check the accuracy of the integral equation methods, we first consider a single orthotropic, elliptic cylindrical inclusion in the unbounded isotropic matrix under uniform remote tensile loading, $\sigma_x^{\mathrm{o}} = \sigma_{\mathrm{o}}$, as shown in Fig. 2. The major and minor axes of the inclusion are denoted by a and b, respectively. Three different aspect ratios ($b/a = 1.0$, 0.75 and 0.50), for the inclusion are considered, and the value of a is assumed to be 70 μm. The elastic constants for the isotropic matrix and the orthotropic inclusion are listed in Table 1 (Davì and Milazzo, 1996). Two different elastic constants for the orthotropic inclusion are considered: in model #1, c_{11} in the inclusion is greater than that in the matrix, and in model #2, c_{11} in the inclusion is smaller than that in the matrix.

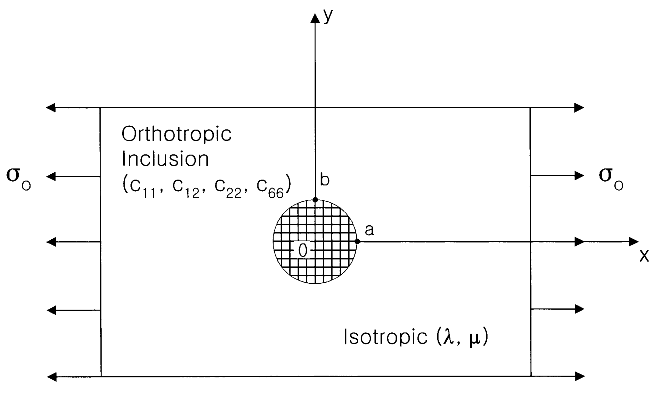


Fig. 2. Geometry of an orthotropic inclusion in unbounded isotropic matrix under uniform remote tensile loading.

Table 1
Material properties of the isotropic matrix and the orthotropic inclusion

Elastic constants (GPa)	Isotropic matrix	Orthotropic inclusion	
		#1	#2
λ	67.34	–	–
μ	37.88	–	–
c_{11}	143.1	279.08	13.95
c_{12}	67.34	7.8	0.39
c_{22}	143.1	30.56	1.53
c_{66}	37.88	11.8	0.59

Figs. 3 and 4 show typical discretized models used in the VIEM and BIEM, respectively. The standard eight-node quadrilateral and six-node triangular elements were used in the VIEM while standard quadratic elements were used in the BIEM. The total number of elements used in VIEM was 144 and in BIEM, it was 80. The number of elements, 144 and 80, was determined based on a convergence test. All computations have been performed on the IBM RS/6000 at UCLA. Table 2 shows the comparison between the analytical solution (Hwu and Yen, 1993; Yang and Chou, 1976) and the numerical solutions using VIEM and BIEM for the normalized tensile stress component (σ_x/σ_x^o) within the orthotropic inclusion under uniform remote tensile loading (σ_x^o). It should be noted that as expected, the stress components inside the inclusion are constant and that there is excellent agreement between the three sets of results for all cases considered.

In order to check the efficiency of the computations, the CPU times required to calculate the stresses in an isotropic matrix with an orthotropic inclusion (#1, $b/a = 1$) using VIEM and BIEM are compared. The measured CPU time in BIEM was 27 s and in VIEM it was 139 s. This is to be expected since BIEM requires discretization of the boundaries only while VIEM requires discretization of the inclusions. This advantage deteriorates as the number of inclusions increases or if the inclusions become inhomogeneous. The number of elements used in the VIEM was determined based on the convergence test. No formal error estimation was carried out; however, the results obtained from analytical method and VIEM were compared in Table 2 to test the accuracy of the VIEM calculations.

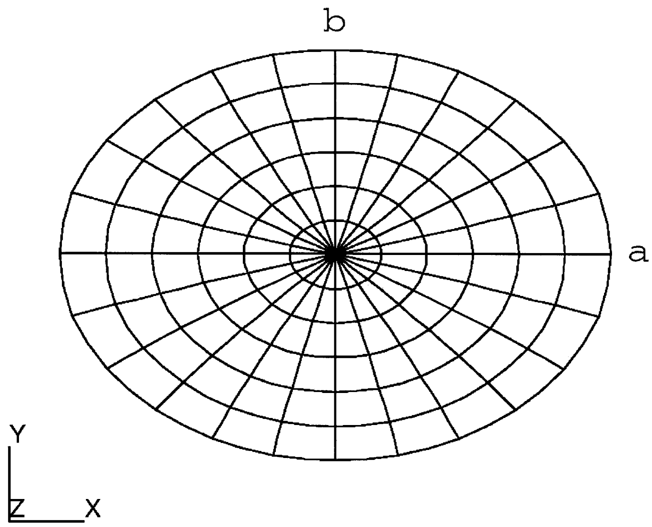


Fig. 3. A typical discretized model in the VIEM.

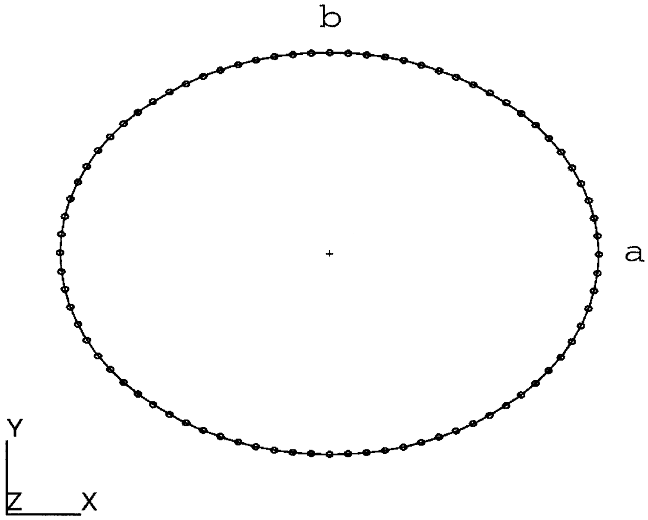


Fig. 4. A typical discretized model in the BIEM.

Table 2
Normalized tensile stress component (σ_x/σ_x^o) within the inclusion due to uniform remote tensile loading (σ_x^o)

	Isotropic matrix with orthotropic inclusion (#1)			Isotropic matrix with orthotropic inclusion (#2)
	$b/a = 1$	$b/a = 0.75$	$b/a = 0.50$	$b/a = 1$
Exact	1.2388	1.3051	1.4206	0.298
VIEM	1.2389	1.3054	1.4213	0.2979
BIEM	1.2397	1.3049	1.4245	0.2986

4. The mixed volume and boundary integral equations

In this section, a mixed volume and boundary integral equation method is introduced as an effective numerical scheme for the solution of plane elastostatic problems in unbounded isotropic matrix containing anisotropic inclusions and voids. The VIEM is effective for problems with anisotropic inclusions due to the fact that only Green's function for the unbounded isotropic matrix is needed in the formulation. However, the method cannot be directly applied to problems involving voids and cracks since the field quantities are undefined in the domain of the integral equation. Problems involving voids can be most effectively solved by BIEM. However, if the material contains a combination of voids and inclusions, it is most advantageous to apply a combination of the two methods. The mixed volume and boundary integral equation for the problem shown in Fig. 5 can be expressed in the form (Lee, 1994)

$$u_m(\mathbf{x}) = u_m^o(\mathbf{x}) - \int_V \delta c_{ijkl} g_{k,l}^{m(\mathrm{M})}(\boldsymbol{\xi}, \mathbf{x}) u_{i,j}(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi} + \int_S c_{ijkl}^{(M)} g_{k,l}^{m(\mathrm{M})} u_i n_j \,\mathrm{d}S(\boldsymbol{\xi}), \tag{26}$$

where V is the volume of the inclusion, S is the surface of the void and $\mathbf{n}$ is the outward unit normal to S. The superscript (M) implies that the quantities involved are for the matrix.

We consider a single orthotropic cylindrical inclusion (orthotropic #1) and a single cylindrical void, each of radius a, in an unbounded isotropic matrix under uniform remote tensile loading as shown in Fig. 5. The distance between the orthotropic inclusion and the void was chosen as $3a$. Fig. 6 shows a typical discretized

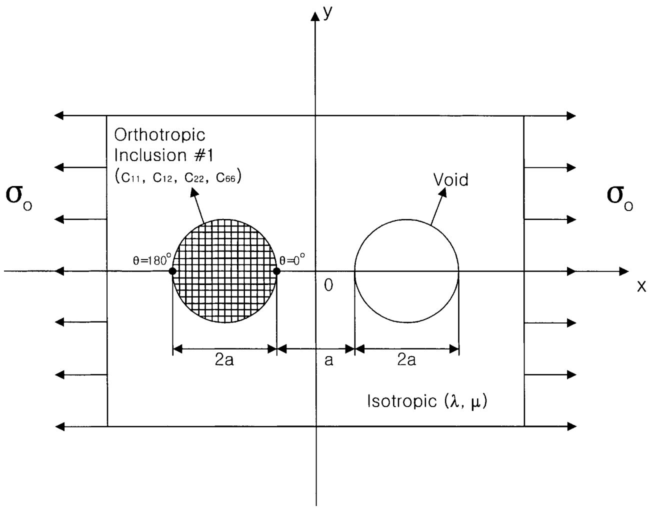


Fig. 5. Geometry of an orthotropic cylindrical inclusion and a cylindrical void in an unbounded isotropic matrix under uniform remote tensile loading.

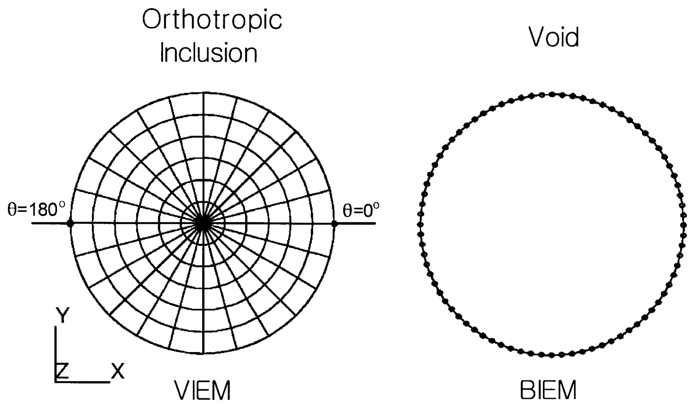


Fig. 6. Discretized model of the problem shown in Fig. 5 in the mixed volume and boundary integral equation method.

model for the problem used in the mixed volume and boundary integral equation method. The number of standard eight-node quadrilateral and six-node triangular elements inside the orthotropic inclusion was 144 while the number of standard quadratic elements on the surface of the void was 80. Assuming that m and n are the number of nodes in V and on S, respectively, the following system of algebraic equations for plane strain problems in the 1–2 plane can be constructed:

$$\begin{bmatrix} [E]_{[(m+n)\times(m+n)]} & [F]_{[(m+n)\times(m+n)]} \\ [G]_{[(m+n)\times(m+n)]} & [H]_{[(m+n)\times(m+n)]} \end{bmatrix} \begin{Bmatrix} \{u_1\}_{(m)} \\ \{u_1\}_{(n)} \\ \{u_2\}_{(m)} \\ \{u_2\}_{(n)} \end{Bmatrix} = \begin{Bmatrix} -\{u_1^0\}_{(m)} \\ -\{u_1^0\}_{(n)} \\ -\{u_2^0\}_{(m)} \\ -\{u_2^0\}_{(n)} \end{Bmatrix}, \tag{27}$$

where

$$[E]_{[(m+n)\times(m+n)]} = \begin{bmatrix} [[VV]-[I]]_{(m\times m)} & [VB]_{(m\times n)} \\ [BV]_{(n\times m)} & [[BB]-\frac{1}{2}[I]]_{(n\times n)} \end{bmatrix},$$

$$[F]_{[(m+n)\times(m+n)]} = \begin{bmatrix} [VV]_{(m\times m)} & [VB]_{(m\times n)} \\ [BV]_{(n\times m)} & [BB]_{(n\times n)} \end{bmatrix},$$

$$[G]_{[(m+n)\times(m+n)]} = \begin{bmatrix} [VV]_{(m\times m)} & [VB]_{(m\times n)} \\ [BV]_{(n\times m)} & [BB]_{(n\times n)} \end{bmatrix},$$

$$[H]_{[(m+n)\times(m+n)]} = \begin{bmatrix} [[VV]-[I]]_{(m\times m)} & [VB]_{(m\times n)} \\ [BV]_{(n\times m)} & [[BB]-\frac{1}{2}[I]]_{(n\times n)} \end{bmatrix},$$

where the symbols $[VV]$, $[VB]$, $[BV]$ and $[BB]$ indicate the field inside the inclusion, the interaction field between the inclusion and the surface of the void, the interaction field between the surface of the void and the inclusion, and the field on the surface of the void, respectively. For the problem sketched in Fig. 6, $m = 433$ and $n = 160$.

The nodal displacement components (u_1, u_2) inside the inclusion and on the surface of the void can be obtained from Eq. (27); the displacements and stresses everywhere can then be calculated without difficulty. It should be noted that it is necessary to use the direct numerical integration scheme in this mixed integral equation formulation.

In order to validate the code based on the mixed volume and boundary integral equation method, the numerical solution of the same problem was also obtained by means of the VIEM only, following the procedure outlined by Lee and Mal (1995, 1997) to represent the void by means of an inclusion with vanishing moduli. Fig. 7 shows a typical discretized model used; the total number of the standard eight-node quadrilateral and six-node triangular elements inside the orthotropic inclusion and inside the "void" was 384. Fig. 8 shows the comparison between the mixed volume and BIEM solution and the VIEM solution for the normalized tensile stress component (σ_x/σ_x^o) at the interface between the isotropic matrix and the orthotropic inclusion under uniform remote tensile loading (σ_x^o). It can be seen that there is excellent agreement between the two sets of results.

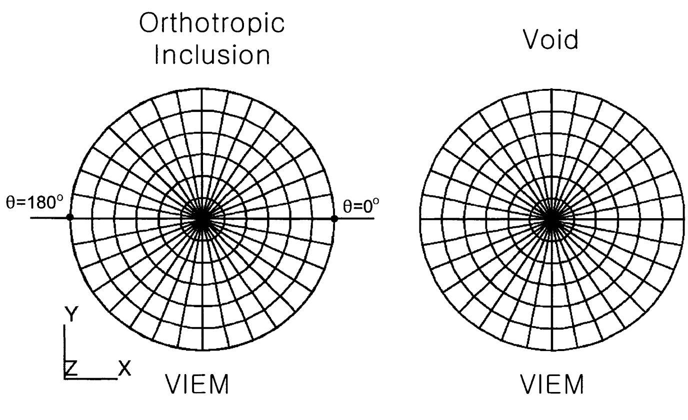
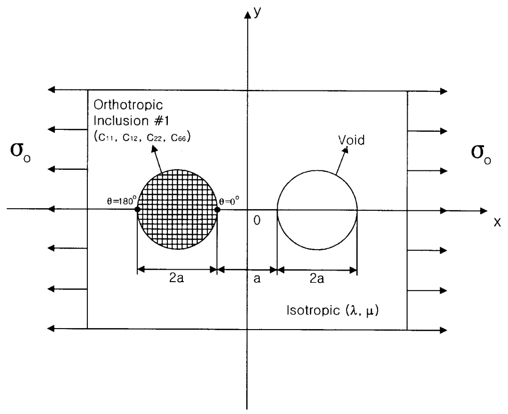


Fig. 7. Discretized model of the problem shown in Fig. 5 in the VIEM.



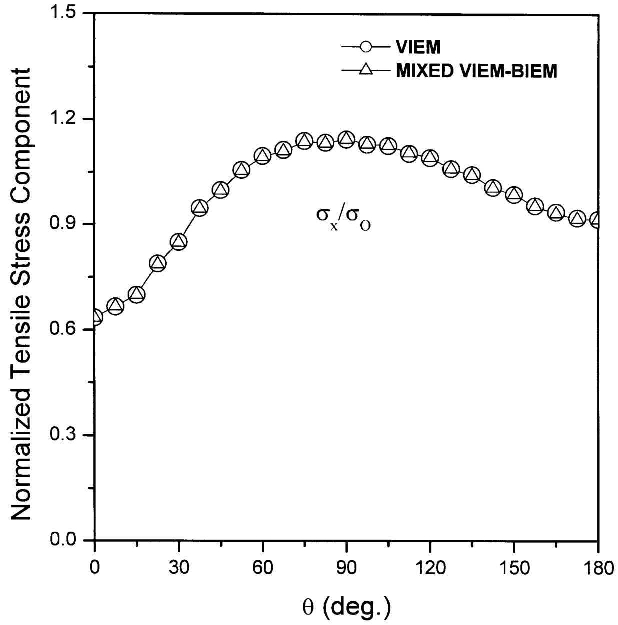


Fig. 8. Normalized tensile stress component (σ_x/σ_x^o) at the interface between the isotropic matrix and the orthotropic inclusion under uniform remote tensile loading (σ_x^o).

5. Concluding remarks

The VIEM is applied to the solution of plane elastostatic problems in unbounded isotropic matrix containing orthotropic inclusions under uniform remote loading. A mixed volume and boundary integral equation method is introduced for the solution of elastostatic problems in unbounded isotropic materials containing anisotropic inclusions as well as voids or cracks. The main advantage of these techniques over those based on finite elements is that they require discretization of the inclusions and void (or crack) surfaces only in contrast to the need to discretize the entire domain. They are similar to the BIEM except for the presence of the volume integral over the inclusions instead of the surface integrals over the two sides of the interface. If the medium contains a small number of (isotropic) inclusions and voids, these methods may not have any advantage over BIEM. However, in the presence of multiple non-smooth inclusions, the numerical treatment in BIEM becomes cumbersome. Since the standard finite elements are used in the VIEM, it is very easy and convenient to handle multiple non-smooth inclusions. In elastodynamic problems in the presence of multiple anisotropic inclusions, the BIEM becomes extremely difficult since closed form expressions for the elastodynamic Green's functions for 2D and 3D for anisotropic media are not available. Two major drawbacks of the methods, at least for the present, are the lack of professionally developed, user-friendly software, and their inability to handle problems involving large deformations.

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References

Banerjee, P.K., 1993. The Boundary Element Methods in Engineering. McGraw-Hill, UK.

Cerrolaza, M., Alarcon, E., 1989. A bi-cubic transformation for the numerical evaluation of the Cauchy principal value integrals in boundary methods. International Journal for Numerical Methods in Engineering 28, 987–999.

Chen, T., 1993. Thermoelastic properties and conductivity of composites reinforced by spherically anisotropic particles. Mechanics of Materials 14, 257–268.

Davì, G., Milazzo, A., 1996. Stress fields in composite cross-ply laminates. In: Ertekin, R.C., Brebbia, C.A., Tanaka, M., Shaw, R. (Eds.), Computational Mechanics, Eleventh International Conference on Boundary Element Technology, BETECH 96, pp. 175–184.

Doblare, M., Espiga, E., Gracia, L., Alcantud, M., 1990. Study of crack propagation in orthotropic materials by using the boundary element method. Engineering Fracture Mechanics 37 (5), 953–967.

Duan, Z.P., Kienzler, R., Herrmann, G., 1986. An integral equation method and its application to defect mechanics. Journal of Mechanics and Physics of Solids 34, 539–561.

Hwu, C., Yen, W.J., 1993. On the anisotropic elastic inclusions in plane elastostatics. Journal of Applied Mechanics–Transactions of the ASME 60, 626–632.

Johnson, W.C., Earmme, Y.Y., Lee J.K, 1980. Approximation of the strain field associated with an inhomogeneous precipitate. I: Theory. Journal of Applied Mechanics–Transactions of the ASME 47, 775–780.

Kushch, V.I., 1998. Interacting cracks and inclusions in a solid by multipole expansion method. International Journal of Solids and Structures 35, 1751–1762.

Lee, J.K., 1994. Micromechanics of crack extension in composites. Ph.D. Dissertation, University of California, Los Angeles.

Lee, J.K., Mal, A.K., 1995. A volume integral equation technique for multiple scattering problems in elastodynamics. Applied Mathematics and Computation 67, 135–159.

Lee, J.K., Mal, A.K., 1997. A volume integral equation technique for multiple inclusion and crack interaction problems. Journal of Applied Mechanics–Transactions of the ASME 64, 23–31.

Lee, K.J., Mal, A.K., 1990. A boundary element method for plane anisotropic elastic media. Journal of Applied Mechanics–Transactions of the ASME 57, 600–606.

Li, H.B., Han, G.M., 1985. A new method for evaluating singular integrals in stress analysis of solids by the direct boundary element method. International Journal for Numerical Methods in Engineering 21, 2071–2098.

Lu, S., Ye, T.Q., 1991. Direct evaluation of singular integrals in elastoplastic analysis by the boundary element method. International Journal for Numerical Methods in Engineering 32, 295–311.

Mal, A.K., Knopoff, L., 1967. Elastic wave velocities in two component systems. Journal of Institute of Mathematics and Applications 3, 376–387.

Mal, A.K., Yang, R.B., 1994. The influence of the fiber-matrix interfacial properties on wave characteristics in metal matrix composites. In: Thompson, D.O., Chimenti, D.E. (Eds.), Review of Progress in Quantitative Nondestructive Evaluation, vol. 13. Plenum Press, New York, pp. 1453–1460.

McPedran, R.C., Movchan, A.B., 1994. The Rayleigh multipole method for linear elasticity. Journal of Mechanics and Physics of Solids 42, 711–727.

Moschovidis, Z.A., Mura, T., 1975. Two-ellipsoidal inhomogeneities by the equivalent inclusion method. Journal of Applied Mechanics–Transactions of the ASME 42, 847–852.

Rizzo, F.J., Shippy, D.J., Rezayat, M., 1985. A boundary integral equation method for radiation and scattering of elastic waves in three dimensions. International Journal for Numerical Methods in Engineering 21, 115–129.

Snyder, M.D., Cruse, T.A., 1975. Boundary integral equation analysis of cracked anisotropic plates. International Journal of Fracture 11 (2), 315–328.

Tomlin, G.R., Butterfield, R., 1974. Elastic analysis of zoned orthotropic continua. Proceedings of ASCE, vol. EM3, pp. 511–529.

Yang, H.C., Chou, Y.T., 1976. Generalized plane problems of elastic inclusions in anisotropic solids. Journal of Applied Mechanics–Transactions of the ASME 43, 424–430.